3.15 $(1,0) \to (-1,0)$

ⓐ $(\cos(t), \sin(t))$ $0 \le t \le \pi$
$(\cos(0), \sin(0)) = (1,0)$ start
$(\cos(\pi), \sin(\pi)) = (-1,0)$ end.

ⓑ $\alpha(t) = (1-t, 0)$ $0 \le t \le 2$
$\alpha(0) = (1,0)$, $\alpha(2) = (-1,0)$

V as 1-form $(x^2-y)\,dx + (-x+2y)\,dy = V^*$

$d(V^*) = -dy \wedge dx - 1\,dx \wedge dy = dx \wedge dy - dx \wedge dy$
$= 0$.

$\therefore$ V is conservative.

Since all 3 paths go from $(1,0)$ to $(-1,0)$,
$\int_\alpha V \cdot ds$ should be the same for all 3.

$$(x^2-y)\,dx + (-x+2y)\,dy = d(f)$$

$$f(x,y) = \frac{x^3}{3} - yx + C(y)$$

$$f_y = 0 - x + C'(y)$$

$C'(y) = 2y$
$C(y) = y^2$

$$\boxed{f(x,y) = \frac{x^3}{3} - yx + y^2}$$

$\nabla f = V$

$$\int_\alpha V \cdot ds = \int_\alpha \nabla f \cdot ds$$

$$= \int_\alpha df = f(-1,0) - f(1,0)$$

$$= \left(\frac{(-1)^3}{3} - 0 + 0\right) - \left(\frac{1}{3} + 0 + 0\right)$$

$$= -\frac{1}{3} - \frac{1}{3} = \boxed{-\frac{2}{3}}.$$

Latest theorem. Stokes' Theorem.

R = set/region

∂R = boundary of R

$$\int_{\partial R} \underbrace{\omega}_{\text{differential form}} = \int_R d\omega$$

Case: ω is 1-form: Green's Thm & Stokes' Thm.

$$\int_{\partial R} V \cdot ds = \int_R \overbrace{(\nabla \times V)}^{\text{curl}(V)} \cdot N \, dA$$

vector version.

Case: ω is a 2-form.

$$\int_{\partial R} \overbrace{\underline{V \cdot N \, dA}}^{\omega} = \int_R \overbrace{(\text{div} V)\underbrace{(d\,\text{Vol})}_{dx \wedge dy \wedge dz}}^{d\omega}$$

divergence theorem.

Example 3.21 Set up $\oint_\alpha P \cdot ds$ two different ways, where α is the ellipse $\frac{x^2}{4}+y^2=1$, oriented clockwise, $P=(x-y, 2x^2)$.

(a) line integral.

$$\oint_\alpha P \cdot ds = \int_\alpha (x-y)\,dx + 2x^2\,dy$$

Parametrize ellipse: $\alpha(t) = (2\cos(t), \sin(t))$

$x = 2\cos t$ $\quad 0 \le t \le 2\pi$

$y = \sin t$ $\quad$ counterclockwise

$dx = x'(t)\,dt = -2\sin(t)\,dt$

$dy = y'(t)\,dt = \cos(t)\,dt$

$$\Rightarrow \oint_\alpha P \cdot ds = -\int_0^{2\pi} (2\cos(t)-\sin(t))(-2\sin(t)\,dt) + 2(2\cos(t))^2(\cos(t)\,dt)$$

$$= \boxed{\int_0^{2\pi} -(4\cos(t)\sin(t) + 2\sin^2(t) + 8\cos^3(t))\,dt}$$

Since α is a closed curve around the inside of the ellipse, we can also use Stoke's Thm.

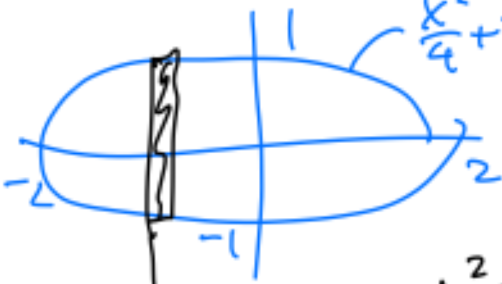
$$\int_{\alpha\,\text{clockwise}} (x-y)\,dx + 2x^2\,dy = -\iint_{E=\text{inside of ellipse}} d\left((x-y)\,dx + 2x^2\,dy\right)$$

$$= -\iint_E \overbrace{d(x-y)}^{dx-dy} \wedge dx + d(2x^2) \wedge dy = -\iint_E -dy \wedge dx + 4x\,dx \wedge dy$$

$4x\,dx$

$$= -\iint +1\,dx\wedge dy + 4x\,dx\wedge dy = -\iint (1+4x)\,\underbrace{dx\wedge dy}_{dx\,dy}$$

$\frac{x^2}{4}+y^2=1$



$$y^2 = 1-\frac{x^2}{4}$$

$$y = \pm\sqrt{1-\frac{x^2}{4}}$$

$$-\sqrt{1-\frac{x^2}{4}} \le y \le \sqrt{1-\frac{x^2}{4}}$$

$$-2 \le x \le 2$$

integrate y first

$$\boxed{= -\int_{x=-2}^{2}\int_{y=-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-x^2/4}} (1+4x)\,dy\,dx}$$

FYI we can use a polar version to make this work

$$\left(\frac{x}{2}\right)^2+y^2=1$$

$$\frac{x}{2} = r\cos\theta$$

$$y = r\sin\theta$$

$$dx = 2r\cos\theta$$

$$dy = r\sin\theta$$

$$0 \le r \le 1$$

$$0 \le \theta \le 2\pi$$

$$dx\wedge dy = 2r\,dr\wedge d\theta$$

Integral would be

$$\boxed{-\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(1+\overset{4x}{8r\cos\theta})\,2r\,dr\,d\theta}$$

Integrating over Surfaces.

① Differential Form (2-forms)

function

$$\iint_S A\,dx\wedge dy + B\,dy\wedge dz + C\,dz\wedge dx$$

z direction — x direction — y direction

vector field

$$= \iint_S \underbrace{(B,C,A)}_{V}\cdot N\,dA$$

vector form

unit normal to surface

$$\iint_S V\cdot N\,dA$$

"flux of V through S" ← This measures the amount the vector field is flowing through S.

How do we set up the integral to do it?

$$\iint_S A\,dx\wedge dy + B\,dy\wedge dz + C\,dz\wedge dx$$

Need to parametrize the surface S

<u>Example</u>: S is the graph of a function, i.e. $z = f(x,y)$

$$\left.\begin{aligned} x &= x \\ y &= y \\ z &= f(x,y) \end{aligned}\right\} \Rightarrow \begin{aligned} dx &= dx \\ dy &= dy \\ dz &= (f_x\,dx + f_y\,dy) \end{aligned}$$

Integrate over the region in the xy plane.

<u>Example</u>: Compute $\iint_S x\,dx\wedge dz + (2y+x^2)\,dy\wedge dz$

where S is the paraboloid $z = 2x^2+2y^2$ restricted to (x,y) in the box $-1 \le x \le 2$, $0 \le y \le 3$.

<u>Parametrize</u>:

$$\begin{aligned} x &= x \\ y &= y \\ z &= 2x^2+2y^2 \end{aligned} \Rightarrow \begin{aligned} dx &= dx \\ dy &= dy \\ dz &= 4x\,dx + 4y\,dy \end{aligned}$$

$$\text{Answer} = \int_{y=0}^{3}\int_{x=-1}^{2} x\,dx\wedge(4x\,dx + 4y\,dy) + (2y+x^2)\,dy\wedge(4x\,dx+4y\,dy)$$

$$= \int_{y=0}^{3} \int_{x=-1}^{2} x(4y)\, dx \wedge dy + (2y + x^2)(4x)\, dy \wedge dx$$

$$4xy\, dx \wedge dy \qquad (-8xy - 4x^3)\, dx \wedge dy$$

$$= \int_{y=0}^{3} \int_{x=-1}^{2} (-4xy - 4x^3)\, \underbrace{dx \wedge dy}_{dx\,dy}$$

$$= \ldots$$

BTW → we just calculated something.